On algebraic variants of the LWE problem

Damien Stehlé

Based on joint works with M. Rosca, A. Sakzad, R. Steinfeld and A. Wallet
Figures borrowed from M. Rosca and A. Wallet

ENS de Lyon, Bitdefender, U. Monash

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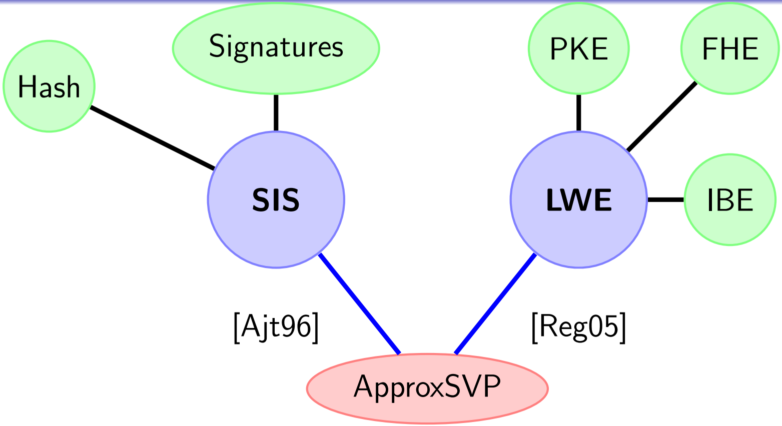
What is this talk about

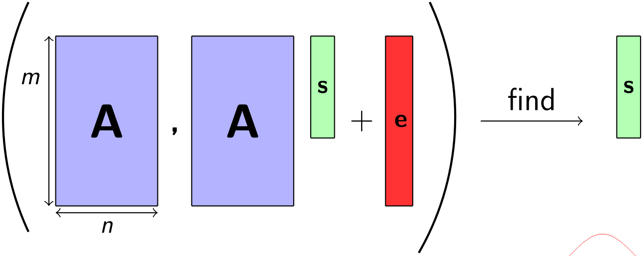


SIS and LWE are lattice problems that are convenient for cryptographic design.

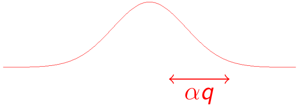
We'll focus on "efficient" variants of LWE.

LWE [Reg05]

LWE parameters: $m \geq n \geq 1$, $q \geq 2$ and $\alpha > 0$.



- $\mathbf{A} \hookleftarrow U(\mathbb{Z}_q^{m \times n})$,
- $\mathbf{s} \hookleftarrow U(\mathbb{Z}_q^{n})$,
- $\mathbf{e} \hookleftarrow D_{\alpha q}^{m}$.



Gaussian error distribution $D_{\alpha q}$

Typical parameters: n proportional to the bit-security,
$q = n^{\Theta(1)}$, $m = \Theta(n \log q)$, $\alpha \approx \sqrt{n}/q$.

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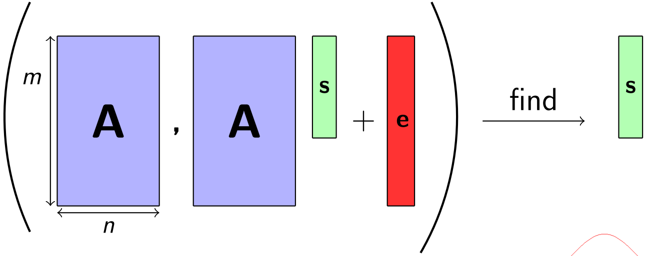

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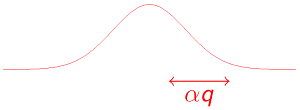

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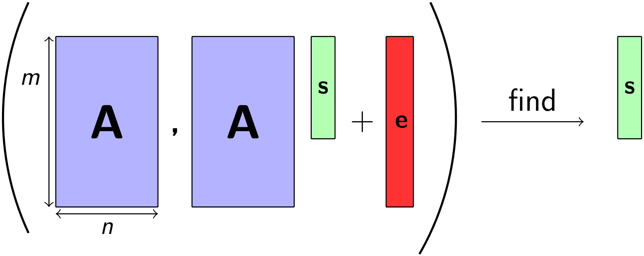
Search LWE as a Closest Vector Problem variant



- **A** defines the Construction-A lattice
$$L_q(\mathbf{A}) = \mathbf{A}\mathbb{Z}_q^n + q\mathbb{Z}^m.$$
- $\mathbf{A}\mathbf{s} + \mathbf{e} \bmod q$ is a point near that lattice.
- Finding $\mathbf{s}$ is finding the closest vector in $L_q(\mathbf{A})$.

LWE is CVP for a uniformly sampled Construction-A lattice, a random lattice vector and a Gaussian lattice offset.

Decision LWE

Decide whether a given $(\mathbf{A}, \mathbf{b})$ is

- uniformly sampled or
- of the form $(\mathbf{A}, \mathbf{A}\mathbf{s} + \mathbf{e})$ with $\mathbf{A}$ and $\mathbf{s}$ uniform and $\mathbf{e}$ sampled from $D_{\alpha q}^m$.

- This is a distribution distinguishing problem.
- More convenient for cryptographic design.
- There are poly-time reductions between search-LWE and decision-LWE [Re05,MiMo11].

[During the talk, I will focus on the search variant]

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Hardness results on LWE (for $\alpha q \geq 2\sqrt{n}$)

The Approximate Shortest Vector Problem

ApproxSIVP$_\gamma$: Given $\mathbf{B} \in \mathbb{Z}^{n \times n}$ defining L,
find $(\mathbf{b}_i)_{i \leq n}$ in L lin. indep. such that $\max \|\mathbf{b}_i\| \leq \gamma \cdot \lambda_n(L)$.

Regev's worst-case to average-case reduction

For q prime and $\leq n^{\mathcal{O}(1)}$, there is a **quantum** poly-time reduction from **ApproxSIVP$_\gamma$** in dimension n to LWE$_{n,m,q,\alpha}$, with $\gamma \approx n/\alpha$.

Best known attack for most parameter ranges: lattice reduction.

$$\text{Time} \approx \exp\left(\frac{n \log q}{\log^2 \alpha} \cdot \log\left(\frac{n \log q}{\log^2 \alpha}\right)\right)$$

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- Representing an LWE instance is quadratic in the bit-security.
- One then performs (at least) matrix-vector multiplications...

Frodo: submission to the NIST post-quantum standardization process
public-key and ciphertexts $\approx$ 10 kB
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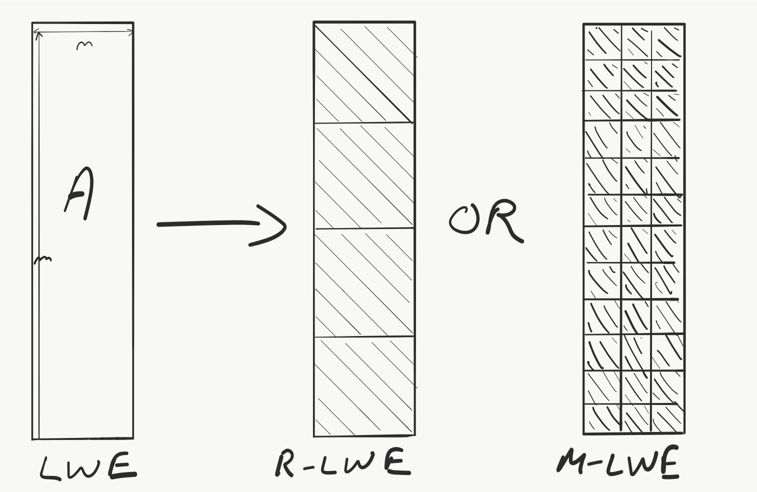
Road-map

- The Learning With Errors problem
- **Algebraic variants of the LWE problem**
- On Polynomial-LWE and Ring-LWE
- The Middle-Product-LWE problem

Take structured matrices!



Polynomial-LWE [SSTX09]

Let $q \geq 2$, $\alpha > 0$, $f \in \mathbb{Z}[x]$ monic irreducible of degree n.

Search P-LWE^f

Given $(a_1, \ldots, a_m)$ and $(a_1 \cdot s + e_1, \ldots, a_m \cdot s + e_m)$, find s.

- s uniform in $\mathbb{Z}_q[x]/f$
- All a_i's uniform in $\mathbb{Z}_q[x]/f$
- The coefficients of the e_i's are sampled from $D_{\alpha q}$

This is LWE, with matrix $\mathbf{A}$ made of stacked blocks $\mathrm{Rot}_f(a_i)$.

The j-th row of $\mathrm{Rot}_f(a_i)$ is made of the coefficients of $x^{j-1} \cdot a_i \bmod f$.

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Hardness of P-LWE

[SSTX09] - oversimplified

For any f monic irreducible, there is a quantum reduction from ApproxSVP$_\gamma$ **for ideals of $\mathbb{Z}[x]/f$** to search P-LWE^f. The error rate α is proportional to γ and

$$\mathrm{EF}(f) := \max_{i<2n} \|x^i \bmod f\|.$$

- This is an adaptation of Regev's ac-wc reduction
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The reduction may be vacuous if ApproxSVP for ideals of $\mathbb{Z}[x]/f$ is easy

- For large approx. factors and some f's, faster algorithms are known for such lattices

  [see Léo's talk]

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Ring-LWE [LPR10]

Let $q \geq 2$, $\alpha > 0$, $f \in \mathbb{Z}[x]$ monic irreducible of degree n.

K: number field defined by f.
$\mathcal{O}_K$: its ring of integers. $\mathcal{O}_K{}^\vee$: its dual ideal.
$\sigma_1, \ldots, \sigma_n$: the Minkowski embeddings.

As complex embeddings come by pairs of conjugates,

the σ_k's give a bijection σ from $K_\mathbb{R} = K \otimes_\mathbb{Q} \mathbb{R}$ to $\mathbb{R}^n$.

Search Ring-LWE^f

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Ring-LWE variants

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- Decision Ring-LWE: distinguish uniform (a_i, b_i)'s from (a_i, b_i)'s as above
- Primal Ring-LWE: replace all ${\mathcal{O}_K}^\vee$'s by $\mathcal{O}_K$.

One may do subtle things with the noise distributions.

Here, we'll be happy if the $\sigma(e_i)$'s are small.

Hardness of Ring-LWE

LPR10 : For all f, there is a reduction from ApproxSVP for $\mathcal{O}_K$-ideals to search Ring-LWE^f.
For f cyclotomic, there is a reduction from search to decision Ring-LWE^f.

PRS17 : For all f, there is a reduction from ApproxSVP for $\mathcal{O}_K$-ideals to decision Ring-LWE^f.

Are there weaker f's for Ring-LWE^f?

- Such potential f's identified in [EHL14,ELOS15,CLS15,CLS16]
- But weakness only with small errors [CIV16a,CIV16b,Pei16]

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A messy landscape...

At least 6 problem families:

- P-LWE^f, search and decision
- R-LWE^f, search and decision
- primal-R-LWE^f, search and decision

Plus Module-LWE^f, a trade-off between these and LWE

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- Is there a relationship between $*$-LWE^f and $*$-LWE^g?
- Can we find one ring to rule them all?

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Do we care?

These algebraic variants do lead to efficient schemes:

NIST p.-q. submissions: Ding, HILA5, KINDI, Kyber, LAC, LIMA, Lizard, Newhope, Saber

Somewhere between 5 and 10 times better than LWE-based Frodo

Most of these use $f = x^n + 1$ with f a power of 2.

For this f, the six problems are identical, and the results have been known for almost 10 years.

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- **On Polynomial-LWE and Ring-LWE**

Joint work with M. Rosca and A. Wallet, Eurocrypt 2018.

- The Middle-Product-LWE problem

From dual to primal

A useful lemma from [LPR10]

Let $t \in ({\mathcal{O}_K}^\vee)^{-1}$ with $t{\mathcal{O}_K}^\vee$ coprime to (q). Then '$\times t$' is an $\mathcal{O}_K$-module isomorphism from ${\mathcal{O}_K}^\vee / q{\mathcal{O}_K}^\vee$ to $\mathcal{O}_K / q\mathcal{O}_K$.

If we have a R-LWE sample $(a_i, b_i = a_i \cdot s + e_i)$,
we can multiply the right hand side by t.

We get $(a_i', b_i') = (a_i, a_i(ts) + (te_i))$.

- ts is now uniform in $\mathcal{O}_K / q\mathcal{O}_K$
- This is a primal R-LWE sample, with noise term $e_i' = te_i$

But is e_i' small? It is if t is small.

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Make the noise small!

Why aren't we happy with possibly large multiplier t?

- We map a CVP instance for a lattice and a quad-form, to an instance for another lattice and another quad-form.
- If we let the quad-form 'free', then all CVP instances can be expressed with the $\mathbb{Z}^m$ lattice.

Goal

Show that there exists $t \in ({\mathcal{O}_K}^\vee)^{-1}$ with $t{\mathcal{O}_K}^\vee$ coprime to (q)

- We consider the Gaussian distribution over $({\mathcal{O}_K}^\vee)^{-1}$
- We show that short vectors are not all trapped in a $({\mathcal{O}_K}^\vee)^{-1} \cdot J$, for a divisor J of (q).
- Tools: inclusion-exclusion and lattice smoothing

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From primal R-LWE to P-LWE

We are given $(a_i, a_i \cdot s + e_i)$ with

- a_i and s in $\mathcal{O}_K$
- e_i with small Minkowski embeddings

We want a related $(a_i', a_i's' + e_i')$ with

- a_i' and s' in $\mathbb{Z}[x]/f$
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Handling the algebra

- $\mathcal{O} := \mathbb{Z}[x]/f$ is an order of $\mathcal{O}_K$.
- Sometimes, they are the same!

The conductor ideal

$\mathcal{C}_{\mathcal{O}} = \{x \in K : x\mathcal{O}_K \subseteq \mathcal{O}\}$ is an $\mathcal{O}_K$-ideal and an $\mathcal{O}$-ideal.

If (q) and $\mathcal{C}_{\mathcal{O}}$ are coprime,
if $t \in \mathcal{C}_{\mathcal{O}}$ is such that $t\mathcal{C}_{\mathcal{O}}^{-1}$ and (q) are coprime,
then "$\times t$" is a ring isomorphism from $\mathcal{O}_K/q\mathcal{O}_K$ to $\mathcal{O}/q\mathcal{O}$.

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Handling the geometry

Relation between the embeddings

For $e \in \mathbb{R}[x]/f$, computing the Minkowski embedding is multiplying the coefficient vector by

$$\mathbf{V}_f = \begin{bmatrix} 1 & \alpha_1 & \alpha_1^2 & \dots & \alpha_1^{n-1} \\ 1 & \alpha_2 & \alpha_2^2 & \dots & \alpha_2^{n-1} \\ \vdots & & \dots & & \vdots \\ 1 & \alpha_n & \alpha_n^2 & \dots & \alpha_n^{n-1} \end{bmatrix},$$

where the α_j's are the roots of f.

We want to know if a noise that has small Minkowski embedding also has small coefficients.

Goal: Show that $\|\mathbf{V}_f^{-1}\|$ is small.

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Root separation

$\|\mathbf{V}_f^{-1}\|$ can be large only if the roots α_j of f are close.

[This can be $2^{\Omega(n)}$, even when f has small coeffs [BM04].]

(1) $f := x^n - c \in \mathbb{Z}[x]$ is great.

(2) Let $P = \sum_{i=1}^{n/2} p_i x^i \in \mathbb{Z}[x]$.

Perturbation: $g := f + P$

For 'small' P, the roots don't move much.

Theorem (Rouché)

If $|P(z)| < |f(z)|$ on a circle, then f and $f + P$ have the same numbers of zeros inside this circle.

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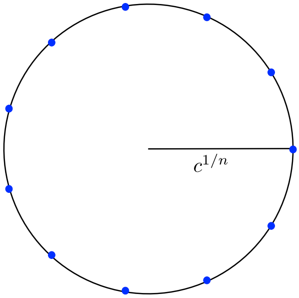

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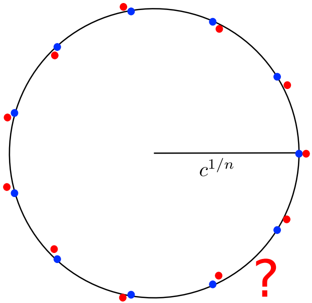

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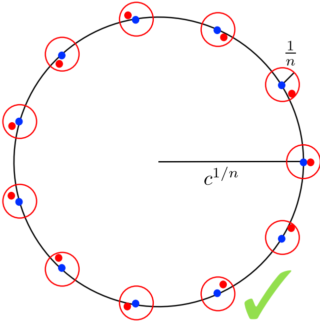

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Road-map

- The Learning With Errors problem
- Algebraic variants of the LWE problem
- On Polynomial-LWE and Ring-LWE
- **The Middle-Product-LWE problem**

Joint work with M. Rosca, A. Sakzad and R. Steinfeld, Crypto 2017.

Middle product

Let $a \in \mathbb{Z}[x]$ of degree $< n$ and $s \in \mathbb{Z}[x]$ of degree $< 2n-1$.

- Their product has $3n-2$ non-trivial coefficients.
- We define $a \circ_n s$ as the middle n coefficients.

$$a \odot_n s := \left\lfloor \frac{(a \cdot b) \bmod x^{2n-1}}{x^{n-1}} \right\rfloor .$$

MP was studied in computer algebra for accelerating computations on polynomials and power series [Sho99,HQZ04].

MP-LWE

Let $q \geq 2$, $\alpha > 0$.

Search MP-LWE

Given $(a_1, \ldots, a_m)$ and $(a_1 \odot_n s + e_1, \ldots, a_m \odot_n s + e_m)$, find s.

- s uniform in $\mathbb{Z}_q[x]$ of degree $< 2n - 1$.
- All a_i's uniform in $\mathbb{Z}_q[x]$ of degree $< n$
- The coefficients of the e_i's are sampled from $D_{\alpha q}$

Titanium: A NIST candidate based on MP-LWE

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Hardness of MP-LWE

$$\text{P-LWE}^{f}_{m,q,\alpha} \text{ reduces to } \text{MP-LWE}_{q,\beta}$$

for **any** monic $f \in \mathbb{Z}[x]$ s.t.

- $\deg(f) = n$
- $\gcd(f_0, q) = 1$
- β grows linearly with α and $\mathrm{EF}(f)$

[This extends [Lyu16] from the SIS to the LWE setup]

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Proof sketch

$$\mathrm{Rot}_f(b) = \mathrm{Rot}_f(a) \times \mathrm{Rot}_f(s) + \mathrm{Rot}_f(e)$$

Take first column

$$M_f \; b = \mathrm{Rot}_f(a) \times M_f \; s + M_f \; e$$

Decompose $\mathrm{Rot}_f(a)$

$$b' = \mathrm{Toep}(a) \; \mathrm{Rot}_f(1) \times M_f \; s + M_f \; e$$

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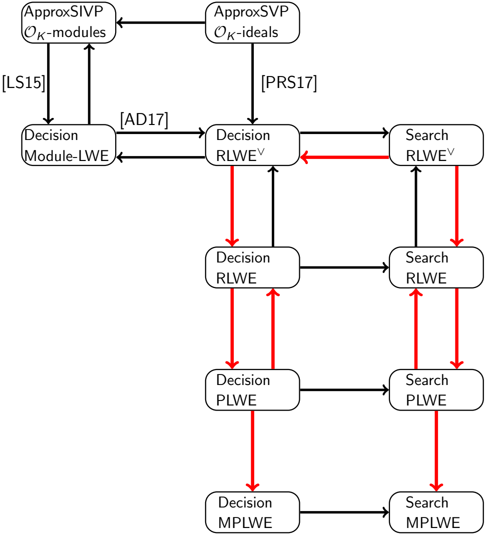
Landscape overview



The search to decision reduction for RLWE$^\vee$ relies on [PRS17] and a left-over hash lemma over ${\mathcal{O}_K}^\vee / q{\mathcal{O}_K}^\vee$.

Some reductions

- are non-uniform
- require small EF
- require small $\|\mathbf{V}_f\|$

Open problems

$\Rightarrow$ Clean the landscape further.

$\Rightarrow$ Relate PLWE^f to PLWE^g.

$\Rightarrow$ Get a search to decision reduction for MP-LWE.

$\Rightarrow$ Get a reduction from MP-LWE to P-LWE.

$\Rightarrow$ Better understand MP-LWE.